

# Congestion network problems and related games

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## Abstract

This paper analyzes network problems with congestion effects from a cooperative game theoretic perspective. It is shown that for network problems with convex congestion costs, the corresponding games have a non-empty core. If congestion costs are concave, then the corresponding game has not necessarily core elements, but it is derived that, contrary to the convex congestion situation, there always exist optimal tree networks. Extensions of these results to a class of relaxed network problems and associated games are derived.

**Keywords:** Congestion, network problems, cooperative games, core.

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# 1 Introduction

Generally speaking, in economic congestion situations agents use facilities from a common pool. Typically the costs of a facility will depend on the number of users. Within game theoretic literature the first paper to consider congestion effects is probably Rosenthal (1973). Here congestion effects are analyzed in a strategic setting. This line of work has been continued by Monderer and Shapley (1996) establishing a connection between potential games and congestion situations. An excellent survey of the related literature can be found in Voorneveld (1999).

Rather surprisingly, in cooperative game theoretic literature congestion effects have been considered far less explicitly. One branch of cooperative literature especially suited by its very nature to accommodate considerations regarding congestion is the literature on Operations Research Games as surveyed by Borm, Hamers, and Hendrickx (2001). An exception is Matsubayashi, Umezawa, Masuda, and Nishino (2005), where hub-spoke network systems with congestion effects are studied using cooperative games.

This paper will focus on a particular extension of a standard operations research problem: minimum cost spanning network problems where the total costs of a specific network depend on the actual number of users of the various parts of the network. In short, we consider congestion network problems. E.g. the congestion of road networks fits within this framework. If more vehicles use one particular part of a road the delay on this part increases and hence costs of using this part increases by means of congestion effects. Another example can be found in the study of telecommunication problems. In communication networks telephones are connected to switching centers to establish connections. The usage of connections can be limited by capacities and costs of using these connections depend on the number of users.

In the classical setting, without congestion, this type of problems is known as minimum cost spanning tree problems. An important topic in this literature, is the issue of a fair cost allocation, firstly pointed out by Claus and Kleitman (1973). The study on the associated cooperative games has been initiated by Bird (1976), introducing a cost allocation rule based on the Prim-Dijkstra algorithm (Prim (1957) and Dijkstra (1959)). Other papers dealing with the issue of cost allocation are Granot and Huberman (1981), Feltkamp, Tijs, and Muto (1994) and Kar (2002). Branzei, Moretti, Norde, and Tijs (2003) introduce the  $P$ -value which is based on the Kruskal algorithm (Kruskal (1956)) and leads to a core element in the underlying minimum cost spanning tree game. Henriot and Moulin (1996) consider a

traffic-based cost allocation method in networks. In their model (a telecommunication network with one switching centre) costs that have to be shared are unrelated to usage. This model could easily be extended to a congestion network model. In Suijs (2003) a cost allocation based on the Bird rule for a network with stochastic costs is introduced. In this model costs of a connection consist of construction costs (deterministic) and maintenance costs (stochastic). A cost structure in which there are different types of costs (but deterministic) like construction costs and usage costs easily fits in the model of congestion network problems and typically leads to concave costs.

The aim of this paper is to analyze congestion network problems from a cooperative point of view by focussing not only on finding an optimal network for a specific set of users but also on the problem of how to allocate the associated jointly generated minimal costs in a fair way among the users.

The structure of the paper is as follows. In section 2 we formally introduce congestion network problems. As two specific examples the cases with constant and linear congestion costs respectively are considered. Section 3 analyzes convex congestion network problems and shows that the corresponding transferable utility games have a non-empty core. Concave congestion network problems are studied in section 4. It is seen that the corresponding games can have an empty core, but that there always exist optimal network structures without cycles. This is not the case for convex congestion network problems. Section 5 considers a type of relaxed congestion network problems. It is derived that the main results for both the convex and concave non-relaxed congestion situations carry forward to this setting. Finally section 6 contains some concluding remarks.

## 2 Congestion network problems and games

Formally, a *congestion network problem*<sup>1</sup> is a triple  $T = (N, *, (k_a)_{a \in A_{N^*}})$ , where  $N = \{1, \dots, n\}$  is a set of agents/players,  $*$  is the source and  $N^* := N \cup \{*\}$ . The set  $A_S$  denotes the set of all arcs between pairs of elements in  $S \subset N^*$ , i.e.  $(S, A_S)$  denotes the complete digraph on  $S$ . For each arc  $a \in A_{N^*}$  the function  $k_a : \{0, 1, \dots, n\} \rightarrow \mathbb{R}_+$  is a nonnegative (weakly) increasing cost function which depends on the number of users of  $a$ . We assume that for all  $a \in A_{N^*}$  it holds that  $k_a(0) = 0$ . Elements of  $A_{N^*}$  will be denoted by  $a$  or by  $(i, j)$ , where  $i, j \in N^*$ . The arc  $(i, j)$  denotes the connection between  $i$  and  $j$  in the direction from  $i$  to  $j$ . The cost function

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<sup>1</sup>The term *congestion* is used here in a broad sense: total costs are increasing in the number of users.

of an arc  $(i, j)$ ,  $i, j \in N^*$  is denoted by  $k_{ij}$ . A congestion network problem is symmetric if  $k_{ij} = k_{ji}$  for all  $i, j \in N^*$ .

Let  $T = (N, *, (k_a)_{a \in A_{N^*}})$  be a congestion network problem. An optimal network can be described by  $f : A_{N^*} \rightarrow \{0, \dots, n\}$ . Let  $F$  be the set consisting of all such networks. A network  $f$  assigns to each arc a number of users of this arc. The indegree for a network  $f \in F$  and  $i \in N^*$  is defined by  $\text{indegree}(i) = \sum_{j \in N^* \setminus \{i\}} f((j, i))$ . Similarly the outdegree is defined by:  $\text{outdegree}(i) = \sum_{j \in N^* \setminus \{i\}} f((i, j))$ . For a coalition  $S \in 2^N \setminus \{\emptyset\}$  the collection of all feasible networks connecting the members of  $S$  to the source is given by:

$$\begin{aligned} F_S = \{ & f \in F \mid \text{outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ & \text{outdegree}(i) = \text{indegree}(i) = 0 \text{ for all } j \in N \setminus S, \\ & f(a) \in \{0, \dots, |S|\}, \forall a \in A_{N^*} \}. \end{aligned}$$

The costs of a network  $f \in F_S$  is naturally defined by:

$$k(f) = \sum_{a \in A_{N^*}} k_a(f(a)).$$

The aim of  $S$  is to construct a feasible network such that all its members are connected to the source and total costs are minimized.

A transferable utility cost game consists of a pair  $(N, c)$ , in which  $N = \{1, \dots, n\}$  is a set of players and  $c : 2^N \rightarrow \mathbb{R}$  is a function assigning to each coalition  $S \in 2^N$  a cost of  $c(S)$ . By definition  $c(\emptyset) = 0$ . With each congestion network problem  $T = (N, *, (k_a)_{a \in A_{N^*}})$  one can associate a congestion network game  $(N, c^T)$ , such that  $c^T(S)$  denotes the minimum costs of a network connecting all players of  $S$  to the source:

$$c^T(S) = \min_{f \in F_S} k(f).$$

**Example 2.1** Consider a congestion network problem  $T = (N, *, (k_a)_{a \in A_{N^*}})$ , such that cost functions are symmetric and constant. This means that for all  $i, j \in N^*$  and for all  $m \in \{1, \dots, n\}$  it holds that  $k_{ij}(m) = k_{ji}(m) = k_{ij}(1)$  and  $k_{ij}(0) = 0$ . So for all  $a \in A_{N^*}$  and for all  $m \in \{1, \dots, n\}$ ,  $k_a(m) = k_a(1)$ . It is readily verified that this congestion network problem is equivalent to a minimum cost spanning tree problem.

Each network  $f$  induces a digraph  $(N^*, A^f)$ .  $A^f$  consists of all arcs used by the network  $f$ :

$$A^f = \{a \in A_{N^*} \mid f(a) > 0\}.$$

Let  $f$  be an optimal network for coalition  $N$ . We can assume that the digraph  $(N^*, A^f)$  does not contain a circuit<sup>2</sup>. To see this, assume that  $(N^*, A^f)$  contains a circuit  $C$ . Change the network  $f$  in such a way that the numbers of users of each arc of  $C$  is decreased by 1 and the number of users of all other arcs stay the same. The resulting network is still a feasible network for  $N$ , since  $C$  is a circuit. Because the cost functions are increasing functions and  $f$  is an optimal network for  $N$  the costs of the resulting network should be the same as the costs of  $f$ . This means that one can change the network in a finite number of steps such that all circuits are deleted and the network left is still optimal for  $N$ .

For an arbitrary arc set  $A$  the set  $E(A)$  is the set of all undirected edges induced by  $A$ :  $E(A) = \{\{i, j\} \mid (i, j) \in A \text{ or } (j, i) \in A\}$ . We say that the digraph  $(N, A)$  contains a cycle if the induced undirected graph  $(N, E(A))$  contains a cycle. Similarly the digraph  $(N, A)$  will be called a tree if  $(N, E(A))$  is a tree. The following example shows that optimal networks in a congestion network problem can contain a cycle.

**Example 2.2** *Consider a symmetric congestion network problem, in which there are three players. For the arcs, the costs of one, two and three users respectively are given by:*

$$\begin{aligned} k_{1*} &= (1, 3, 6), \\ k_{2*} &= (5, 10, 15), \\ k_{3*} &= (3, 7, 11), \\ k_{12} &= (3, 6, 9), \\ k_{13} &= (1, 2, 3), \\ k_{23} &= (1, 2, 3). \end{aligned}$$

*The optimal network of  $N$  is drawn in Figure 1.*

The core of a TU-game  $(N, c)$  is given by:

$$\text{Core}(c) = \left\{ x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S), \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

The core of a game consists of those cost allocation vectors such that no coalition has an incentive to split off. If cost functions are constant and

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<sup>2</sup>A digraph  $(N^*, A)$  contains a circuit if there exists a sequence  $(i_1, i_2), (i_2, i_3), \dots, (i_{p-1}, i_p)$  such that  $i_1 = i_p$  and  $(i_m, i_{m+1}) \in A$  for all  $m \in \{1, \dots, p-1\}$ .

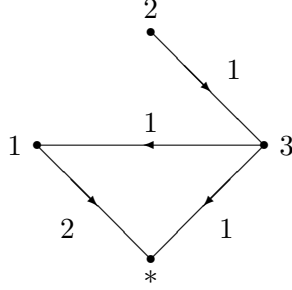


Figure 1: Optimal network of the problem given in Example 2.2.

symmetric (see Example 2.1), then the congestion network game is a minimum cost spanning tree game and the core will be non-empty (cf. Bird (1976)). This does not hold for arbitrary cost functions, which is illustrated in the following example.

**Example 2.3** Consider a congestion network problem  $T = (N, *, (k_a)_{a \in A_{N*}})$ , with  $N = \{1, 2, 3\}$  and symmetric functions  $k_a$  defined by:

$$\begin{aligned} k_{1*} &= (1, 1, 2), \\ k_{2*} &= (1, 1, 2), \\ k_{3*} &= (1, 1, 2), \\ k_{12} &= (0, 0, 0), \\ k_{13} &= (0, 0, 0), \\ k_{23} &= (0, 0, 0). \end{aligned}$$

The game  $(N, c^T)$  is given by:

$S$	1	2	3	12	13	23	$N$
$c^T$	1	1	1	1	1	1	2

For an element  $x \in \text{Core}(c^T)$  the following equations should hold:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_1 + x_2 &\leq 1 \\ x_1 + x_3 &\leq 1 \\ x_2 + x_3 &\leq 1. \end{aligned}$$

Adding the last three equations yields a contradiction with the first one. Hence  $\text{Core}(c^T) = \emptyset$ .

A cost function  $k_a$  is linear if for all  $m \in \{0, \dots, n\}$   $k_a(m) = m \cdot k_a(1)$ . Congestion network problems with linear costs are very similar to congestion network problems with constant costs.

A path from  $i$  to  $*$  in the digraph  $(N^*, A_{N^*})$  is a sequence of arcs  $P = ((i_0, i_1), (i_1, i_2), \dots, (i_{p-1}, i_p))$ , such that  $i_0 = i$ ,  $i_p = *$  and  $i_k \neq i_l$  for all  $k, l \in \{0, \dots, p\}$ . It is intuitively clear that if each player  $i \in N$  chooses a path  $P_i$  to the source such that the costs of this path (which equal  $\sum_{a \in P_i} k_a(1)$ ) is minimal, the combination of these paths yields an optimal structure for the problem corresponding to coalition  $N$ .

Consider a relaxation of the problem in which all arcs are fully public. Now for each coalition  $S \in 2^N$  one can consider the relaxed problem of finding a network with minimal costs, connecting all players of  $S$  to the source. Coalition  $S$  is allowed to use any arc to establish this. Hence the set of feasible networks for  $S \in 2^N$  becomes:

$$\begin{aligned} \{f \in F \mid & \text{outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ & \text{outdegree}(i) - \text{indegree}(i) = 0 \text{ for all } i \in N \setminus S, \\ & f(a) \in \{0, \dots, |S|\}, \forall a \in A_{N^*}\}. \end{aligned}$$

Denote the corresponding coalitional value by  $d^T(S)$ . Clearly, for each coalition an optimal network in this relaxed problem is given by the combination of the optimal paths of its members. Hence the game is additive and assigning to each player the costs of its optimal path will yield the unique core element. Since  $c^T(N) = d^T(N)$  and  $c^T(S) \geq d^T(S)$  for all  $S \in 2^N$  this allocation also yields a core element of the original congestion network game  $c^T$ .

**Example 2.4** Consider the symmetric congestion network problem  $T$  with linear cost functions of Figure 2 with  $N = \{1, 2, 3\}$ . The numbers at an arc represent  $k_a(1)$ . The game  $c^T$  is given by:

$S$	1	2	3	12	13	23	$N$
$c^T(S)$	7	10	5	12	16	15	21

The optimal path of player 1 is  $P_1 = ((1, *))$ , for players 2 and 3 the optimal paths are:  $P_2 = ((2, 1), (1, *))$  and  $P_3 = ((3, *))$ . Each player  $i \in N$  pays the costs of  $P_i$  which yields the allocation vector  $x = (7, 9, 5)$ . It can be easily checked that this is indeed a core element of the game  $(N, c^T)$ .

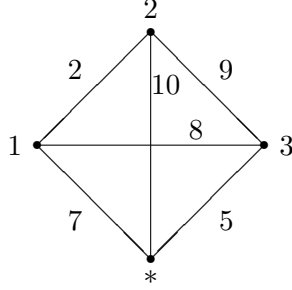


Figure 2: Congestion network problem of Example 2.4 with linear cost functions.

### 3 Convex congestion

In this section we examine so called *convex congestion network problems*  $T = (N, *, (k_a)_{a \in A_{N*}})$  in which all  $k_a$  are convex. It will be shown that for a corresponding convex congestion game there always exists a core allocation.

A cost function  $k_a$ ,  $a \in A_{N*}$  is convex if for all  $m \in \{1, \dots, n-1\}$ :

$$k_a(m+1) - k_a(m) \geq k_a(m) - k_a(m-1).$$

As illustrated in Example 2.2 convex congestion network problems can have an optimal structure containing a cycle. On the other hand all congestion network games with convex cost functions have a non-empty core, as is stated in the following theorem.

**Theorem 3.1** *Let  $T = (N, *, (k_a)_{a \in A_{N*}})$  be a convex congestion network problem, then  $\text{Core}(c^T) \neq \emptyset$ .*

To prove this theorem we will consider a specific relaxation, in which non integer networks are allowed. A network in this relaxed problem can be described by:  $f : A_{N*} \rightarrow [0, n]$ . Let  $\mathcal{F}$  be the set consisting of all relaxed networks. The set of all feasible networks connecting a coalition  $S \in 2^N$  to the source is extended to:

$$\begin{aligned} \mathcal{F}_S = \{ f \in \mathcal{F} \mid & \text{outdegree}(i) - \text{indegree}(i) = 1 \text{ for all } i \in S, \\ & \text{outdegree}(i) = \text{indegree}(i) = 0 \text{ for all } j \in N \setminus S \}. \end{aligned} \quad (1)$$

One can extend the cost functions  $k_a$ ,  $a \in A_{N*}$  in a piecewise linear way



to  $\bar{k}_a : [0, n] \rightarrow \mathbb{R}^+$ :<sup>3</sup>

$$\bar{k}_a(x) := k_a(\lfloor x \rfloor) + (x - \lfloor x \rfloor) \cdot (k_a(\lceil x \rceil) - k_a(\lfloor x \rfloor)).$$

Clearly  $\bar{k}_a$  is convex on  $[0, n]$ . The cost of a network  $f \in \mathcal{F}_S$  is now defined by:

$$\bar{k}(f) = \sum_{a \in A_{N^*}} \bar{k}_a(f(a)).$$

Moreover, let the relaxed congestion network game  $(N, \bar{c}^T)$  be defined by:

$$\bar{c}^T(S) := \min_{f \in \mathcal{F}_S} \bar{k}(f)$$

for all  $S \in 2^N \setminus \{\emptyset\}$ . Note that:

$$\bar{c}^T(S) \leq c^T(S),$$

since for all  $S \in 2^N \setminus \{\emptyset\}$  it holds that  $F_S \subset \mathcal{F}_S$  and  $\bar{k}_a$  extends  $k_a$ . The following lemma proves that extending a congestion network problem in a linear way does not change the values of the corresponding game.

**Lemma 3.1** *Let  $T = (N, *, (k_a)_{a \in A_{N^*}})$  be a congestion network problem. Then  $\bar{c}^T(S) = c^T(S)$  for all  $S \subset N$ .*

**Proof:** Take  $S \in 2^N$ . It is sufficient to prove that there exists a network  $f^* \in \mathcal{F}_S$  such that  $\bar{c}^T(S) = \bar{k}(f^*)$  and  $f^*$  is integer valued. Let  $f \in \mathcal{F}_S$  such that  $\bar{c}^T(S) = \bar{k}(f)$  and let  $f$  be such that for all other  $g \in \mathcal{F}_S$  such that  $\bar{c}^T(S) = \bar{k}(g)$  it holds that:

$$|\{a \in A_{S^*} \mid f(a) \notin \mathbb{N}\}| \leq |\{a \in A_{S^*} \mid g(a) \notin \mathbb{N}\}|.$$

This means that  $f$  is chosen within the set of all optimal networks for  $S$  such that the number of arcs that have a non-integer  $f$ -value is minimal.

Let  $D$  be the set of arcs that have a non-integer  $f$ -value. If  $D = \emptyset$  we are done, since then  $f \in F_S$ . So assume  $D \neq \emptyset$ . Because the difference between outdegree and indegree of a node in  $S^*$  is integer, each node is either adjacent to no arcs of  $D$ , or to two or more. So it is possible to find a cycle  $C$  inside  $D$ .

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<sup>3</sup>For  $x \in \mathbb{R}$  the lower entier function is defined by:  $\lfloor x \rfloor = \max\{y \in \mathbb{Z} \mid y \leq x\}$ , similarly the upper entier function is defined by:  $\lceil x \rceil = \min\{y \in \mathbb{Z} \mid y \geq x\}$ .

Choose one of the two orientations of  $C$ . Let  $C^+$  be the set of arcs in  $C$  that are directed according to this orientation. Let  $C^- = C \setminus C^+$ . Perturb the network  $f$  in two opposite directions. First, consider the network  $f^+$ , which is obtained network  $f$  by increasing the number of users through  $C^+$  by  $\varepsilon$  and decreasing the number of users through  $C^-$  by  $\varepsilon$ .

$$f^+(a) := \begin{cases} f(a) & \text{if } a \in A_{N^*} \setminus C, \\ f(a) + \varepsilon & \text{if } a \in C^+, \\ f(a) - \varepsilon & \text{if } a \in C^-. \end{cases}$$

Similarly, define  $f^-$  by increasing the number of users through  $C^-$  by  $\varepsilon$  and decreasing the number of users through  $C^+$  by  $\varepsilon$ . The value of  $\varepsilon$  is chosen maximal such that the following inequalities are true for all  $a \in C$ :

$$\begin{aligned} \lfloor f(a) \rfloor &\leq f^+(a) \leq \lceil f(a) \rceil, \\ \lfloor f(a) \rfloor &\leq f^-(a) \leq \lceil f(a) \rceil. \end{aligned}$$

It is left to the reader to verify that  $f^+$  and  $f^-$  are both non-negative and that the net indegree remains the same for all nodes. We conclude that  $f^+$  and  $f^-$  are feasible networks for  $S$ .

Network  $f$  is the average of the two feasible networks  $f^+$  and  $f^-$ . Because of the definition of  $\varepsilon$  and the partially linearity of  $\bar{k}_a$  for all arcs, we have:

$$\bar{k}(f) = \bar{k}(\tfrac{1}{2}f^- + \tfrac{1}{2}f^+) = \tfrac{1}{2}\bar{k}(f^-) + \tfrac{1}{2}\bar{k}(f^+).$$

Because  $f$  has minimal costs, this can only be the case if  $f^-$  and  $f^+$  have the same costs as  $f$ .

By definition of  $\varepsilon$  there is at least one arc  $\hat{a}$  such that  $f^-(\hat{a})$  or  $f^+(\hat{a})$  is integer valued, say  $f^+(\hat{a})$ . This gives that the number of non-integer  $f^+$ -valued arcs is strictly less than the number of integer  $f$ -valued arcs, contradicting the assumption that  $f$  is minimal with respect to this feature.  $\square$

The next lemma proves that the relaxed congestion network game  $(N, \bar{c}^T)$  has a non-empty core if the cost functions are convex. A characterization of games with a non-empty core is given independently by Bondareva (1963) and Shapley (1967). This characterization uses the notion of balanced sets and maps. A map  $\lambda : 2^N \setminus \{\emptyset\} \rightarrow \mathbb{R}_+$  is called balanced if:

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) e^S = e^N.$$

The vector  $e^S$  denotes the characteristic vector for coalition  $S$ :

$$e_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \in N \setminus S. \end{cases}$$

A collection  $B$  of coalitions is a balanced collection if there exists a balanced map  $\lambda$  such that:

$$B = \{S \in 2^N \setminus \{\emptyset\} \mid \lambda(S) > 0\}.$$

A cost game  $(N, c)$  is balanced if for each balanced map  $\lambda$  it holds that:

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S) c(S) \geq c(N).$$

Bondareva (1963) and Shapley (1967) proved that a game is balanced if and only if it has a non-empty core.

**Lemma 3.2** *Let  $T = (N, *, (k_a)_{a \in A_{N*}})$  be a convex congestion network problem. Then  $(N, \bar{c}^T)$  is a balanced game.*

**Proof:** Let  $\lambda$  be a balanced map and  $B$  the balanced collection such that:

$$B = \{S \in 2^N \setminus \{\emptyset\} \mid \lambda(S) > 0\}.$$

For all  $S \in B$  take a network  $f^S \in \mathcal{F}_S$  such that  $\bar{c}^T(S) = \bar{k}(f^S)$ . Define the network  $f \in \mathcal{F}$  as:

$$f(a) := \sum_{S \in B} \lambda(S) f^S(a).$$

Let  $i \in N$ . The difference between outdegree and indegree according to  $f$  equals:

$$\begin{aligned} \text{outdegree}^f(i) - \text{indegree}^f(i) &= \sum_{S \in B} \lambda(S) (\text{outdegree}^{f^S}(i) - \text{indegree}^{f^S}(i)) \\ &= \sum_{S \in B: i \in S} \lambda(S) (\text{outdegree}^{f^S}(i) - \text{indegree}^{f^S}(i)) \\ &= \sum_{S \in B: i \in S} \lambda(S) \cdot 1 \\ &= 1. \end{aligned}$$

In the last equality it is used that  $\lambda$  is a balanced map. Since  $f$  is also nonnegative it holds that  $f \in \mathcal{F}_N$ . Moreover:

$$\begin{aligned}
\bar{k}(f) &= \sum_{a \in A_{N^*}} \bar{k}_a(f(a)) \\
&= \sum_{a \in A_{N^*}} \bar{k}_a\left(\sum_{S \in B} \lambda(S) f^S(a)\right) \\
&\leq \sum_{a \in A_{N^*}} \sum_{S \in B} \lambda(S) \bar{k}_a(f^S(a)) \\
&= \sum_{S \in B} \sum_{a \in A_{N^*}} \lambda(S) \bar{k}_a(f^S(a)) \\
&= \sum_{S \in B} \lambda(S) \bar{k}(f^S) \\
&= \sum_{S \in B} \lambda(S) \bar{c}^T(S).
\end{aligned}$$

Note that for an arc  $a \in A_{N^*}$ ,  $f(a)$  is a convex combination of  $f^S(a)$  for  $S \in B$  and 0. So the above inequality follows from the convexity of the functions and the fact that  $\bar{k}_a(0) = 0$ . Since  $f \in \mathcal{F}_N$ :

$$\begin{aligned}
\bar{c}^T(N) &\leq \bar{k}(f) \\
&\leq \sum_{S \in B} \lambda(S) \bar{c}^T(S),
\end{aligned}$$

proving that  $(N, \bar{c}^T)$  is a balanced game.  $\square$

Theorem 3.1 is now a direct consequence of Lemma 3.1 and Lemma 3.2.

## 4 Concave congestion

In this section we examine so called *concave congestion network problems*<sup>4</sup>  $T = (N, *, (k_a)_{a \in A_{N^*}})$  in which all  $k_a$  are concave. A cost function  $k_a$ ,  $a \in A_{N^*}$  is concave if for all  $m \in \{1, \dots, n-1\}$ :

$$k_a(m+1) - k_a(m) \leq k_a(m) - k_a(m-1).$$

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<sup>4</sup>The term of *concave congestion* might be somewhat misleading, but it is meant that costs are increasing in the number of users and costs functions are concave. This means that there are scale merits: if the number of users increases the average costs per user decrease.

In a convex congestion network problem the corresponding digraph of an optimal network for the grand coalition could contain cycles. The following theorem proves that concave congestion network problems have optimal network which induce a tree.

**Theorem 4.1** *Let  $T = (N, *, (k_a)_{a \in A_{N^*}})$  be a concave congestion network problem. There exists a network  $f \in F_N$  such that  $k(f) = c^T(N)$  and  $(N^*, A^f)$  is a tree.*

**Proof:** Let  $f \in F_N$  such that  $c^T(N) = k(f)$ , this means that  $f$  is an optimal network for  $N$ . As we have noted before we may assume that  $A^f$  does not contain any circuits. Now suppose that  $A^f$  is not a tree. Then  $(N^*, A^f)$  contains a cycle. Hence there exist two nodes  $i, j \in N^*$  such that there are two disjoint paths from  $i$  to  $j$ . Define  $P_1$  as the set of arcs of the first path and  $P_2$  as the set of arcs of the second path, so  $P_1 \cap P_2 = \emptyset$ . Define:

$$\varepsilon := \min_{a \in P_1 \cup P_2} f(a).$$

Now we perturb the network  $f$  in two different ways. First consider the network  $f^1$ , which is obtained from  $f$  by increasing the number of users through  $P_1$  by  $\varepsilon$  and decreasing the number of users through  $P_2$  by  $\varepsilon$ .

$$f^1(a) = \begin{cases} f(a) & \text{if } a \in A_{N^*} \setminus (P_1 \cup P_2), \\ f(a) + \varepsilon & \text{if } a \in P_1, \\ f(a) - \varepsilon & \text{if } a \in P_2. \end{cases}$$

Note that by the definition of  $\varepsilon$ ,  $f^1$  is nonnegative and integer-valued. The difference between indegree and outdegree does not change for a node. It follows that  $f^1 \in F_N$ . Similarly  $f^2 \in F_N$  arises from  $f$  by decreasing the number of users through  $P_1$  by  $\varepsilon$  and increasing the number of users through  $P_2$  by  $\varepsilon$ . Hence:

$$f^2(a) = \begin{cases} f(a) & \text{if } a \in A_{N^*} \setminus (P_1 \cup P_2), \\ f(a) - \varepsilon & \text{if } a \in P_1, \\ f(a) + \varepsilon & \text{if } a \in P_2. \end{cases}$$

Because  $f$  is an optimal network it holds that:

$$\begin{aligned}
0 &\leq k(f^1) - k(f) \\
&= \sum_{a \in P_1} (k_a(f(a) + \varepsilon) - k_a(f(a))) - \sum_{a \in P_2} (k_a(f(a)) - k_a(f(a) - \varepsilon)) \\
&\leq \sum_{a \in P_1} (k_a(f(a)) - k_a(f(a) - \varepsilon)) - \sum_{a \in P_2} (k_a(f(a) + \varepsilon) - k_a(f(a))) \\
&= k(f) - k(f^2) \\
&\leq 0.
\end{aligned}$$

The second inequality follows from the concavity of the functions  $k_a$ . We can conclude that the costs of the networks  $f$ ,  $f^1$  and  $f^2$  are all the same. This means that  $f^1$  and  $f^2$  are both optimal networks for  $N$ .

By the definition of  $\varepsilon$  in either  $f^1$  and  $f^2$  the number of users of at least one arc in  $A^f$  becomes zero. Assume without loss of generality that this is  $f^1$ . Then  $|A^{f^1}| < |A^f|$ . If  $f^1$  is a tree, then the proof is finished. If not there is again a cycle in  $f^1$  and we can repeat the above reasoning. Note that this process will end in a finite number of steps, because the number of arcs with a positive number of users is decreased by at least one in each step. Hence we will find a network  $\hat{f} \in F_N$  such that  $k(\hat{f}) = k(f)$  and  $(N^*, A^{\hat{f}})$  is a tree.  $\square$

In the following example it is illustrated that a congestion network problem with concave cost functions need not to be balanced.

**Example 4.1** Consider a symmetric congestion network problem with 6 players. Assume that the cost function of arcs which are not drawn in Figure 3, are too expensive to use in any optimal network. The cost functions for arcs towards the source equal:

$$k_{i*} = (10, 20, 20, 20, 20, 20), \quad i \in \{1, 2, 3\}.$$

For all other edges drawn in Figure 3, the cost function is linear with coefficient 10:  $k_a(m) = 10m$ ,  $m \in \{0, 1, \dots, 6\}$ . The value of a coalition can be easily calculated by using Theorem 4.1 and just checking all trees connecting  $S$  to the source. This yields that  $c^T(\{1245\}) = c^T(\{2346\}) = c^T(\{1356\}) = 50$  and  $c^T(N) = 80$ . The collection  $B = \{\{1245\}, \{1356\}, \{2346\}\}$  is a balanced collection with  $\lambda(S) = \frac{1}{2}$  for all  $S \in B$ . It follows that:

$$\frac{1}{2}(c^T(\{1245\}) + c^T(\{1356\}) + c^T(\{2346\})) \leq c^T(N),$$

which proves that  $(N, c^T)$  is not balanced.

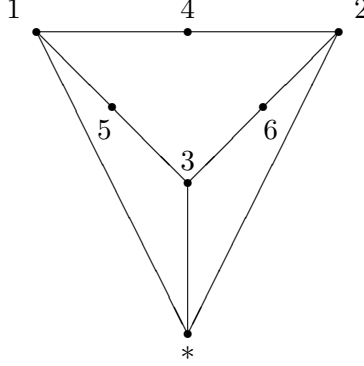


Figure 3: Sketch of the situation of Example 4.1.

## 5 Relaxed congestion network games

A specific type of relaxed congestion network games (using a piecewise linear extension) has been used to prove that convex congestion network games are balanced. The main feature of a relaxed congestion network problem is that the feasible networks need not to be integer valued. Relaxed congestion network problems can be the appropriate model if e.g. it is possible that nodes use different connections to the source for certain periods of time (as e.g. in computer networks). This feature can be easily modelled by dropping the restriction that networks should be integer valued.

In this section we extend the results for convex and concave congestion network problems earlier developed to the relaxed congestion network problems.

A relaxed congestion network problem is given by  $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$  in which  $k_a : [0, n] \rightarrow \mathbb{R}_+$  is a (weakly) increasing cost function for all  $a \in A_{N*}$ . The corresponding relaxed congestion network game is denoted by  $(N, c^T)$ . The set of all networks is denoted by  $\mathcal{F}$ . For  $S \in 2^N$  all feasible networks are denoted by  $\mathcal{F}_S$  (see formula (1)).

A relaxed congestion network problem is convex if all functions  $k_a$  are convex. Similarly a relaxed congestion network problem is concave if all functions  $k_a$  are concave.

If a relaxed congestion network problem  $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$  is given, one can easily find a related congestion network problem by restricting the function  $k_a$  to the domain  $\{0, \dots, n\}$ . The congestion network problem achieved in this way will be denoted by  $T(\mathcal{T})$ . In the following example it is shown that a relaxed congestion network game can differ from a congestion

network game with the restricted cost functions.

**Example 5.1** Consider a symmetric two person congestion network problem as depicted in the first graph of Figure 4, with player set  $N = \{1, 2\}$ . The cost functions are given by:  $k_{*1}(x) = 2x^2$ ,  $k_{*2}(x) = 3x$  and  $k_{12}(x) = 0$ ,  $x \in [0, 2]$ . Note that cost functions are convex in this example. The optimal networks of the relaxed problem and the non-relaxed problem can be found in the second and third picture of Figure 4. It can be calculated that  $c^T(N) = 4\frac{1}{2}$  and  $c^{T(T)}(N) = 5$ .

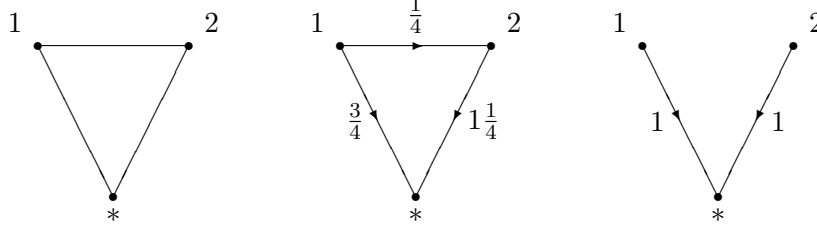


Figure 4: A two person congestion network problem.

Convex congestion network games are balanced games, this result is also true for relaxed convex congestion network games.

**Theorem 5.1** Let  $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$  be a relaxed convex congestion network problem. Then  $\text{Core}(c^T) \neq \emptyset$ .

The proof follows the proof of Lemma 3.2, since the latter does not use the assumption of piecewise linearity of cost functions. Only convexity has been used.

Concave congestion network problems always have an optimal network for  $N$  which induces a tree. This remains true for relaxed concave congestion network problems.

**Theorem 5.2** Let  $\mathcal{T} = (N, *, (k_a)_{a \in A_{N*}})$  be a relaxed concave congestion network problem. Then there is at least one network  $f \in \mathcal{F}_N$  such that  $\bar{k}(f) = c^T(N)$  and  $(N^*, A^f)$  is a tree.

The proof of this theorem is similar to the proof of Theorem 4.1 and is therefore omitted. As a result of Theorem 5.2 it can be proved that restricting a concave cost function does not change the corresponding congestion network game.



**Corollary 5.1** *Let  $\mathcal{T} = (N, *, (k_a)_{a \in A_{N^*}})$  be a relaxed concave congestion network problem. Then  $c^{\mathcal{T}}(S) = c^{\mathcal{T}(\mathcal{T})}(S)$  for all  $S \in 2^N$ .*

**Proof:** Let  $S \in 2^N$ . Clearly Theorem 5.2 implies that there exists a network  $f^S \in \mathcal{F}_S$  such that  $c^{\mathcal{T}}(S) = k(f^S)$  and  $(S^*, A^{f^S})$  is a tree. This implies that  $f^S \in F_S$ . Thus it holds that:

$$\begin{aligned} c^{\mathcal{T}(\mathcal{T})}(S) &\geq k(f^S) \\ &= c^{\mathcal{T}}(S). \end{aligned}$$

Since it always holds that  $c^{\mathcal{T}(\mathcal{T})}(S) \leq c^{\mathcal{T}}(S)$  it follows that  $c^{\mathcal{T}(\mathcal{T})}(S) = c^{\mathcal{T}}(S)$ .  $\square$

## 6 Concluding remarks

In this paper we considered congestion network problems and related games. Costs of using arcs depend on the number of users of these arcs, which is a natural assumption. In case of convex cost functions the corresponding games have a non-empty core. Rather surprisingly, if cost functions are concave (which means that there are merits of scale), the underlying game can have an empty core. It turned out that there is always an optimal tree network in the case of concave cost functions.

Throughout the paper it is implicitly assumed that the demand of each node/player equals one. The model can be extended to a model in which each node has its own demand, not necessarily equal to one. In this setting a player may use several paths to the source to obtain his required demand. Extending the model in this way generates a broader set of problems to which the model applies. For example in flow situations it is natural that each node has its own demand. This extended model can be described by the tuple  $(N, *, (d_i)_{i \in N}, (k_a)_{a \in A_{N^*}})$ , where  $d_i \in \mathbb{R}_+$  denotes the demand of player  $i \in N$ . The definition of a feasible network of a coalition  $S \subseteq N$  becomes:

$$\begin{aligned} F_S = \{ f \in \tilde{F} \mid &\text{outdegree}(i) - \text{indegree}(i) = d_i \text{ for all } i \in S, \\ &\text{outdegree}(i) = \text{indegree}(i) = 0 \text{ for all } j \in N \setminus S, \\ &f(a) \geq 0, \forall a \in A_{N^*} \}, \end{aligned}$$

where  $\tilde{F}$  denotes the set of all networks. With some modifications with respect to the notations and the definitions (especially the definition of a feasible network which is given above) the analysis and results in this paper

will carry through to this setting. To maintain the direct connection to the literature on minimum cost spanning tree problems, we have chosen not to present our results in such a general framework.

## References

- Bird, C. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks*, **6**, 335–350.
- Bondareva, O.N. (1963). Some applications of linear programming methods to the theory of cooperative games (in russian). *Problemy Kibernetiki*, **10**, 119–139.
- Borm, P., H. Hamers, and R. Hendrickx (2001). Operations research games: a survey. *TOP*, **9**, 139–216.
- Branzei, R., S. Moretti, H. Norde, and S. Tijs (2003). The P-value for cost sharing in minimum cost spanning tree problems. CentER DP 2003-129, Tilburg University, Tilburg, The Netherlands. To appear in “Essays on Cooperative Games in Honor of Guillermo Owen,” (Ed. G. Gambarelli), Kluwer Academic Publishers.
- Claus, A. and D.J. Kleitman (1973). Cost allocation of a spanning tree. *Networks*, **3**, 289–304.
- Dijkstra, E. (1959). A note on two problems in connection with graphs. *Numerische Mathematik*, **1**, 269–271.
- Feltkamp, V., S. Tijs, and S. Muto (1994). On the irreducible core and the equal remaining obligations rule of minimum cost spanning extension problems. CentER DP 1994-106, Tilburg University, Tilburg, The Netherlands.
- Granot, D. and G. Huberman (1981). On minimum cost spanning tree games. *Mathematical Programming*, **21**, 1–18.
- Henriet, D. and H. Moulin (1996). Traffic-based cost allocation in a network. *The Rand Journal of Economics*, **27**, 332–345.
- Kar, A. (2002). Axiomatization of the Shapley value on minimum cost spanning tree games. *Games and Economic Behavior*, **38**, 265–277.
- Kruskal, J.B. (1956). On the shortest spanning subtree of a graph and the traveling salesman problem. *Proceedings of the American Mathematical Society*, **7**, 48–50.

- Matsubayashi, N., M. Umezawa, Y. Masuda, and H. Nishino (2005). A cost allocation problem arising in hub-spoke network systems. *European Journal of Operations Research*, **160**, 821–838.
- Monderer, D. and L.S. Shapley (1996). Potential games. *Games and Economic Behavior*, **14**, 111–124.
- Prim, R. (1957). Shortest connection networks and some generalizations. *Bell Systems Technical Journal*, **36**, 1389–1401.
- Rosenthal, R.W. (1973). A class of games possessing pure-strategy Nash equilibria. *Journal of Economic Theory*, **2**, 65–67.
- Shapley, L.S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, **14**, 453–460.
- Suijs, J. (2003). Cost allocation in spanning network enterprises with stochastic connection costs. *Games and Economic Behavior*, **42**, 156–171.
- Voorneveld, M. (1999). *Potential games and interactive decisions with multiple criteria*. Ph. D. thesis, Tilburg University, The Netherlands.